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A NOTE ON COMPACTIFICATIONS OF G-SPACES

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A note on compactifications of G-spaces *)

by

J. de Vries

ABSTRACT

Let $< G,X,\pi >$ be a topological transformation group (ttg) such that all orbit-closures are compact and, in addition, the action is equicontinuous. If X is a Tychonoff space, then $< G,X,\pi >$ can equivariantly be embedded in a ttg $< G,Y,\sigma >$ where Y is a compact Hausdorff space. In particular, for any compact Hausdorff group G, every Tychonoff G-space can equivariantly be embedded in a compact Hausdorff G-space.

KEY WORDS & PHRASES: *G-space, equivariant embedding, compactification*

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*) This paper is not for review; it is meant for publication elsewhere.

1. PRELIMINARY RESULTS

A *topological transformation group* (ttg) or a G-*space* is a triple $< G,X,\pi >$ where G is a topological group, X is a topological space, and π is an action of G on X, i.e. $\pi: G \times X \to X$ is a continuous mapping satisfying the conditions $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ ($x \in X$ and $s,t \in G$; e denotes the identity of G). We refer to [5] as a general reference for ttg's. The notation of [5] will be used with the following modification: function symbols are written as operators from the left. Thus, the *transitions* π^t and the *motions* π_x of a G-space are defined by $\pi^t x := \pi(t,x) =: \pi_x t$ ($t \in G$, $x \in X$).

If we speak of a Tychonoff or of a compact Hausdorff G-space, we mean that the space X (and not G) has the corresponding topological properties. *Unless otherwise stated, we shall always assume that* X *is a Tychonoff space.* Any uniformity in X which is compatible with the topology of X will be called a uniformity *for* X. Since several uniformities for X will be considered, we shall often indicate explicitly the uniformity with respect to which a certain uniform notion is used. Thus, a ttg $< G,X,\pi >$ is called $\mathcal{U}$-bounded* whenever $\mathcal{U}$ is a uniformity for X and $\{\pi_x : x \in X\}$ is an equicontinuous set of functions from G into the uniform space $(X,\mathcal{U})$. Note that $< G,X,\pi >$ is $\mathcal{U}$-bounded if and only if $\{\pi_x : x \in X\}$ is $\mathcal{U}$-equicontinuous at e, that is,

$$\forall \alpha \in \mathcal{U},\ \exists U \in \mathcal{V}_e : (\pi^t x,x) \in \alpha \qquad \text{for all } t \in U,\ x \in X$$

(here $\mathcal{V}_e$ denotes the neighbourhood filter of e in G).

If $< G,X,\pi >$ and $< G,Y,\sigma >$ are G-spaces, then a mapping $f: X \to Y$ is called *equivariant* whenever $f \circ \pi^t = \sigma^t \circ f$ for every $t \in G$. We will be concerned with a special case of the following general problem: *does there exist, for every (Tychonoff)* G*-space* $< G,X,\pi >$, *an equivariant embedding*

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* The term *bounded* occurs in [3] and also in [8] and [9]. In [1] the term *motion equicontinuous* is used.

into a compact Hausdorff G*-space* $< G,Y,\sigma >$? For a categorical setting of the problem, see sub-section 7.3 of [8]. In [1] and [2] one can find results which are closely connected with this question. In particular, in [1] BROOK proved:

THEOREM 1. *Let* $< G,X,\pi >$ *be a ttg and let* $\mathcal{U}$ *be a uniformity for* X *such that*

(i) $< G,X,\pi >$ *is* $\mathcal{U}$*-bounded;*

(ii) *Each* π^t, $t \in G$, *is* $\mathcal{U}$*-uniformly continuous.*

Then X *can equivariantly be embedded in a compact Hausdorff* G*-space.*

In [1], the proof is given by extending each transition to a homeomorphism of the Samuel compactification sX of the uniform space $(X,\mathcal{U})$. In [8], we gave a different proof, which was based on an application of ASCOLI's theorem in the space $C_c(G,sX)$. In that proof we did not need condition (ii). It is not difficult, however, to show here directly that this condition can be omitted from the hypothesis of Theorem 1. The proof is based on the well-known fact that *any Tychonoff* G*-space* X *admits a uniformity* $\mathcal{U}^*$ *such that every transition is* $\mathcal{U}^*$*-uniformly continuous.* The uniformity $\mathcal{U}^*$ can be obtained in the following way. Let $\mathcal{U}$ be any uniformity for X, and let $\mathcal{U}^*$ be the weakest uniformity in X making all transitions $\pi^t: (X,\mathcal{U}^*) \to (X,\mathcal{U})$, $t \in G$, uniformly continuous. Since each π^t is a homeomorphism of X, $\mathcal{U}^*$ is compatible with the topology of X. And since the composition of $\pi^t: (X,\mathcal{U}^*) \to (X,\mathcal{U}^*)$ with any $\pi^s: (X,\mathcal{U}^*) \to (X,\mathcal{U})$ is uniformly continuous, being equal to $\pi^{st}: (X,\mathcal{U}^*) \to (X,\mathcal{U})$, it follows that π^t is $\mathcal{U}^*$-uniformly continuous ($t \in G$).

The following lemma shows that in Theorem 1 the uniformity $\mathcal{U}$ can simply be replaced by the corresponding uniformity $\mathcal{U}^*$.

LEMMA 1. *The Tychonoff* G*-space* X *is* $\mathcal{U}$*-bounded if and only if it is* $\mathcal{U}^*$*-bounded.*

PROOF. "Only if": clear from the inclusion $\mathcal{U} \subseteq \mathcal{U}^*$.
"If": observe that a base of $\mathcal{U}^*$ is constituted by all sets of the form

$$\alpha^*_A := \{(x,y) \in X \times X : (\pi^s x,\pi^s y) \in \alpha \text{ for all } s \in A\}$$

with $\alpha \in \mathcal{U}$ and A a finite subset of G. Now suppose that $< G,X,\pi >$ is $\mathcal{U}$-bounded. Let $\alpha \in \mathcal{U}$ and let A be a finite subset of G. By $\mathcal{U}$-boundedness there is $U \in \mathcal{V}_e$ such that

$$(\pi^{us}x, \pi^{s}x) \in \alpha \qquad \text{for all } u \in U,\ x \in X \text{ and } s \in G.$$

Since A is finite, there is $V \in \mathcal{V}_e$ such that $sV \subseteq Us$ for all $s \in A$. It follows that

$$(\pi^{sv}x, \pi^{s}x) \in \alpha \qquad \text{for all } v \in V,\ x \in X \text{ and } s \in A,$$

that is, $(\pi^{v}x, x) \in \alpha^{*}_{A}$ for all $v \in V$ and $x \in X$. This shows that $< G,X,\pi >$ is $\mathcal{U}^{*}$-bounded. □

COROLLARY. *A Tychonoff* G*-space* X *can equivariantly be embedded in a compact Hausdorff* G*-space if and only if there is a uniformity* $\mathcal{U}$ *for* X *such that* X *is* $\mathcal{U}$*-bounded.*

PROOF. "Only if": a straightforward compactness argument (give X the relative uniformity of the compactification).
"If": Follows from Lemma 1 and Theorem 1. □

2. MAIN RESULTS

Let $< G,X,\pi >$ be a ttg with G compact. The topology of G is generated by a (unique) uniformity $\mathcal{R}$ which coincides with both the left and the right uniformity of G. Let $\mathcal{U}$ be a uniformity for X. Replacing $\mathcal{U}$ by $\mathcal{U}^{*}$, we may assume that each $\pi^{t}: X \to X$ is $\mathcal{U}$-uniformly continuous ($t \in G$). Since G is compact and $\pi: G \times X \to X$ is continuous, it follows from a straightforward compactness argument *that* $\{\pi^{t} : t \in G\}$ *is* $\mathcal{U}$*-equi-uniformly continuous.*

The following results were obtained by HIMMELBERG [7]. Let $(Y,\mathcal{W})$ be a uniform space, Z any set, and $f: Y \to Z$ a mapping. Then there exists a finest uniformity on Z such that f is uniformly continuous (quotient uniformity). If $(f\times f)[\mathcal{W}]$ happens to be a uniformity, then it coincides with this quotient uniformity. If, in addition, $f^{\leftarrow}[z]$ is compact for every $z \in Z$, then the quotient topology on Z relative to f is the topology of the quotient uniformity.

We apply these results to the uniform space $(G\times X, \mathcal{R}\times\mathcal{U})$ and the mapping $\pi: G\times X \to X$, where $<G,X,\pi>$, $\mathcal{R}$ and $\mathcal{U}$ are as above.

<u>LEMMA 2</u>. *The filter* $\mathcal{U}' := (\pi\times\pi)[\mathcal{R}\times\mathcal{U}]$ *in* $X\times X$ *is a uniformity in* X *and it is compatible with the topology of* X.

<u>PROOF</u>. Observe that $\pi: G\times X\to X$ is an open mapping, so that X has the quotient topology relative to π. Moreover, $\overset{\leftarrow}{\pi}[x]$ is a compact subset of $G\times X$ for every $x\in X$, because it is a closed subset of the compact set $G\times\pi_x[G]$. Hence, by the above-mentioned results it is sufficient to prove that $\mathcal{U}'$ is a uniformity. Clearly only the "triangle axiom" needs verification. To this end we introduce the following notation. A base for the uniformity $\mathcal{R}$ of G is formed by the sets $\rho_U := \{(s,t)\in G\times G : st^{-1}\in U\}$ with $U\in\mathcal{V}_e$. For $U\in\mathcal{V}_e$ and $\alpha\in\mathcal{U}$, set

$$[U,\alpha] := (\pi\times\pi)[\rho_U\times\alpha] = \{(x,y)\in X\times X : \exists s,t\in G \text{ with } s^{-1}t\in U \text{ and } (\pi^s x,\pi^t y)\in\alpha\}.$$

We have to show that for every $U\in\mathcal{V}_e$ and $\alpha\in\mathcal{U}$ there exist $V\in\mathcal{V}_e$ and $\beta\in\mathcal{U}$ such that

$$(1)\qquad [V,\beta]\circ[V,\beta]\subseteq[U,\alpha].$$

By assumption, $\{\pi^t : t\in G\}$ is equi-uniformly continuous on X with respect to $\mathcal{U}$, so there exists $\beta\in\mathcal{U}$ such that

$$(2)\qquad (\pi^u\times\pi^u)[\beta]\circ(\pi^v\times\pi^v)[\beta]\subseteq\alpha$$

for all $u,v\in G$. In addition, let $V\in\mathcal{V}_e$ be such that $V^2\subseteq U$. We claim that (1) holds for this choice of $[V,\beta]$. For let $(x,y)\in[V,\beta]\circ[V,\beta]$. Then there exists $z\in X$ such that $(x,z)\in[V,\beta]$ and $(z,y)\in[V,\beta]$, i.e., there exist $u,v,p,q\in G$ such that

$$u^{-1}v \in V \quad \text{and} \quad (\pi^u x, \pi^v z) \in \beta,$$
$$p^{-1}q \in V \quad \text{and} \quad (\pi^p z, \pi^q y) \in \beta.$$

Let $s := v^{-1}u$ and $t := p^{-1}q$. Then $(\pi^s x, z) \in (\pi^{v^{-1}} \times \pi^{v^{-1}})[\beta]$ and $(z, \pi^t y) \in (\pi^{p^{-1}} \times \pi^{p^{-1}})[\beta]$, so by (2),

(3) $$(\pi^s x, \pi^t y) = (\pi^s x, z) \circ (z, \pi^t y) \in \alpha.$$

However, $s^{-1}t = u^{-1}vp^{-1}q \in V^2 \subseteq U$, so (3) implies that $(x,y) \in [U,\alpha]$. □

LEMMA 3. *The mapping* $\pi: (G \times X, \mathcal{R} \times \mathcal{U}') \to (X, \mathcal{U}')$ *is uniformly continuous.*

REMARK. By Lemma 2 and HIMMELBERG's results it is clear that $\pi: (G \times X, \mathcal{R} \times \mathcal{U}) \to (X, \mathcal{U}')$ is uniformly continuous. We want to replace here the uniformity of the domain by the possibly *weaker* uniformity $\mathcal{R} \times \mathcal{U}'$.

PROOF. Let $U \in \mathcal{V}_e$ and $\alpha \in \mathcal{U}$. We have to find $V \in \mathcal{V}_e$ and $\beta \in \mathcal{U}$ such that for all $s,t \in G$ and $x,y \in X$,

$$st^{-1} \in V \ \& \ (x,y) \in [V,\beta] \Rightarrow (\pi^s x, \pi^t y) \in [U,\alpha].$$

Take $W \in \mathcal{V}_e$ such that $W^2 \subseteq U$. There exists $V \in \mathcal{V}_e$ for which $Vw \subseteq wW$ for all $w \in G$ ([6], Theorem 4.9; this is equivalent to saying that the left and the right uniformity coincide on G). Note that $V \subseteq W$.

Consider $s,t \in G$ with $st^{-1} \in V$. In addition, consider $x,y \in X$ with $(x,y) \in [V,\alpha]$. Then there exist $u,v \in G$ with $u^{-1}v \in V$ and $(\pi^u x, \pi^v y) \in \alpha$. Set $p := us^{-1}$ and $q := vt^{-1}$. Then

$$p^{-1}q = su^{-1}vt^{-1} \in sVt^{-1} \subseteq st^{-1}W \subseteq W^2 \subseteq U,$$

and $(\pi^p \pi^s x, \pi^q \pi^t y) = (\pi^u x, \pi^v y) \in \alpha$, so that, indeed, $(\pi^s x, \pi^t y) \in [U,\alpha]$. □

THEOREM 2. *Every Tychonoff* G*-space* X *with* G *a compact Hausdorff topological group can equivariantly be embedded in a compact Hausdorff* G*-space.*

PROOF. Let $\mathcal{U}'$ be as in the Lemmas 2 and 3. Then the ttg $< G,X,\pi >$ is clearly $\mathcal{U}'$-bounded. Now the result follows from the corollary to Lemma 1. □

If G is not assumed to be compact, we can prove the following theorem (of which Theorem 2 is a special case):

THEOREM 3. *Let* $< G,X,\pi >$ *be a ttg such that for every* $x \in X$ *the orbit-closure* $K[x] := cl_X\pi_x[G]$ *is compact. If* $\{\pi^t \,\vdots\, t \in G\}$ *is equicontinuous on* X *with respect to a uniformity* $\mathcal{U}$ *for* X, *then* X *can equivariantly be embedded in a compact Hausdorff* G-*space*.

PROOF. Let E_X be the enveloping semigroup of the ttg $< G,X,\pi >$, i.e. the closure of $\{\pi^t \,\vdots\, t \in G\}$ in X^X. It follows from Theorem 7 in [4], that E_X is a compact Hausdorff topological group. Equicontinuity of $\{\pi^t \,\vdots\, t \in G\}$ implies equicontinuity of E_X, and this, in turn, implies that the evaluation mapping $\delta\colon (h,x) \mapsto h(x)\colon E_X \times X \to X$ is continuous. Clearly, δ is an action of E_X on X, and Theorem 2 implies that the ttg $< E_X,X,\delta >$ can equivariantly be embedded in a ttg $< E_X,Y,\sigma >$ with Y a compact Hausdorff space. Let $f\colon x \to Y$ denote this embedding. An action ζ of G on Y can be defined by $\zeta(t,y) := \sigma(\pi^t,y)$, $t \in G$, $y \in Y$ (observe that the mapping $t \mapsto \pi^t\colon G \to E_X$ is continuous). So $< G,Y,\zeta >$ is a compact Hausdorff G-space. Because

$$f(\pi(t,x)) = f(\delta(\pi^t,x)) = \sigma(\pi^t,fx) = \zeta(t,fx),$$

$t \in G$, $x \in X$, it follows that f is an equivariant embedding of $< G,X,\pi >$ into $< G,Y,\zeta >$. □

REMARKS. As to the equicontinuity of $\{\pi^t \,\vdots\, t \in G\}$ it may be useful to remark that in [4] only equicontinuity of $\{\pi^t|_{K[x]} \,\vdots\, t \in G\}$ is required for every $x \in X$. The following example (which is a modification of the example given in [4]) shows that this condition may not imply equicontinuity of $\{\pi^t \,\vdots\, t \in G\}$ on all of X; nevertheless, the G-space under consideration can equivariantly be embedded in a compact G-space.

EXAMPLE. Let X be the following open subset of the complex plane: $X := \{z \in \mathbb{C} \,\vdots\, a < |z| < b\}$ with $0 < a < b$. Define $\pi\colon \mathbb{R} \times X \to X$ by $\pi(t,z) = z\, exp(it|z|)$. Then each orbit-closure is compact, on each orbit-closure

K[z] the set $\{\pi^t|_{K[z]} : t \in \mathbb{R}\}$ is equicontinuous, but $\{\pi^t : t \in \mathbb{R}\}$ is not equicontinuous on X. However, the action π extends continuously to an action of $\mathbb{R}$ on the compact space $Y := \{z \in \mathbb{C} : a \le |z| \le b\}$ (this is in accordance with Proposition 2 in [8], since X is locally compact).

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